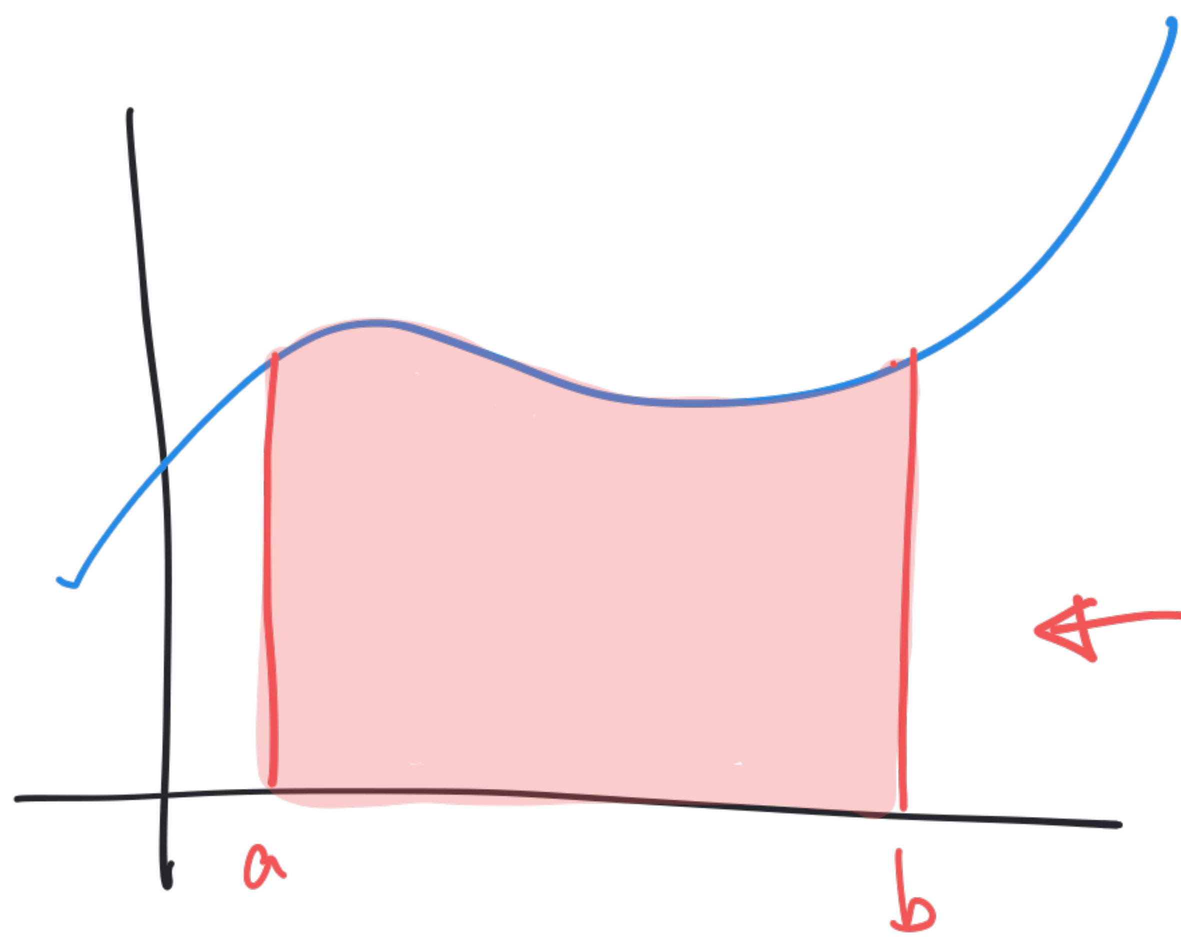
Intro Video: Section 5.2
The Definite Integral

Math F251X: Calculus I

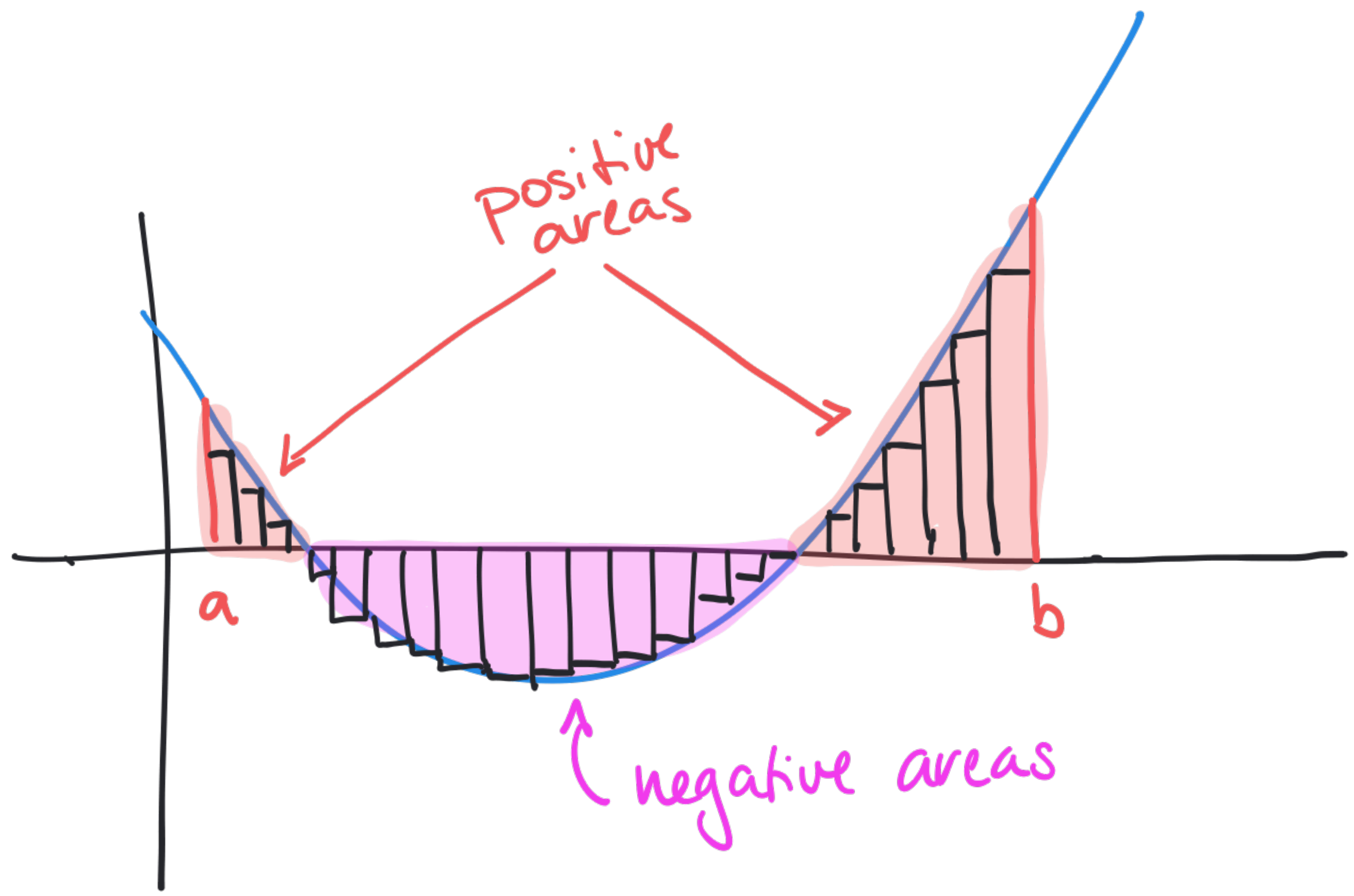
The definite integral



If $f(x) \geq 0$, then the area under the curve is measured as the definite integral

$$\int_a^b f(x)\,dx$$

"the integral from a to b of $f(x)$ dx"

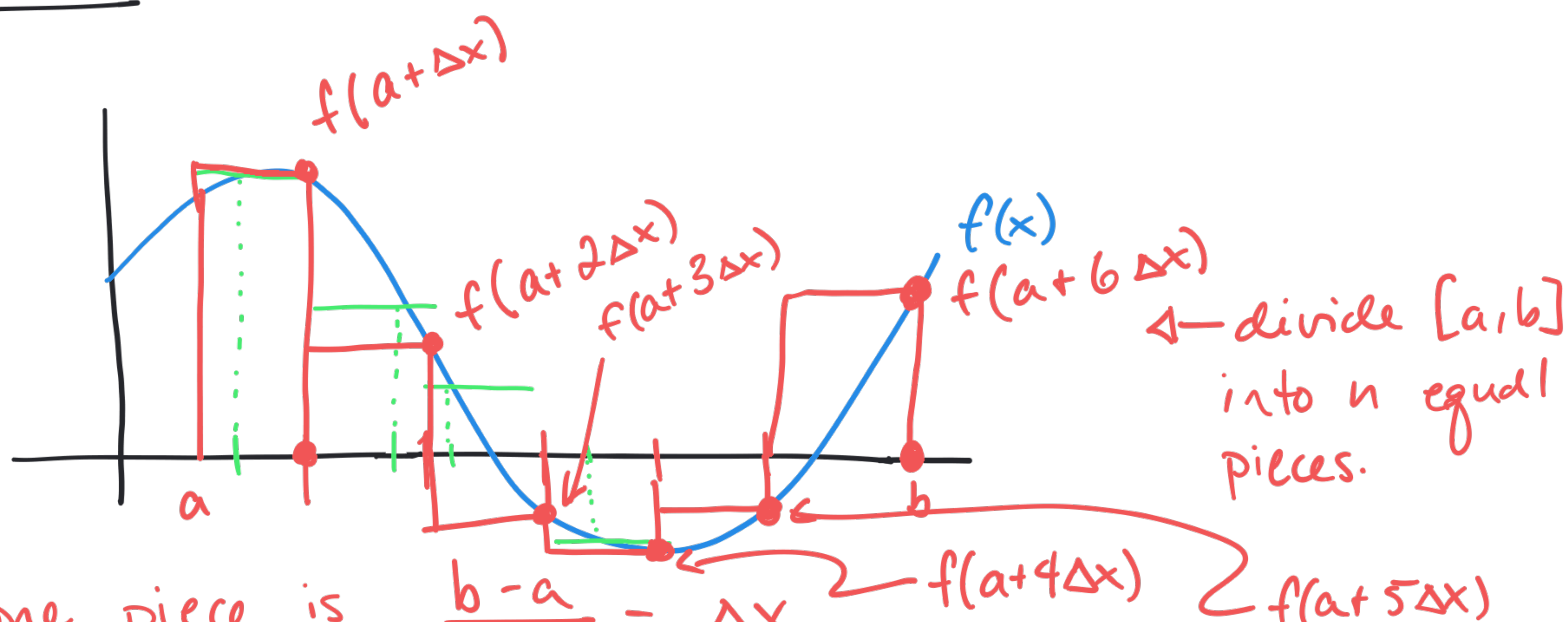


$\int_a^b f(x)\,dx$ measures signed areas between the curve and the x-axis, for x in [a,b].

How to determine $\int_a^b f(x)\,dx$? Chop area into n rectangles and let $n \to \infty$, so that $\int_a^b f(x)dx := \lim_{n\to\infty}$ (area of all the rectangles)

How to write down a limit equal to $\int_a^b f(x)\,dx$:

(a <u>Riemann</u> sum)



Width of one piece is $\frac{b-a}{n} = \Delta x$

We will use right-hand endpoints to determine our rectangle height.

(But deep math says you can use any sample point!)

$$\text{area of all } n \text{ rectangles} = \sum_{i=1}^{n} \underbrace{f(a+i\Delta x)}_{\text{height } i^{\text{th}} \text{ rect}} \cdot \underbrace{\Delta x}_{\text{width}}$$

Def'n: Given $f(x)$ that is sufficiently nice ("integable") [for us, f is continuous or has a finite number of holes] on $[a,b]$, we define

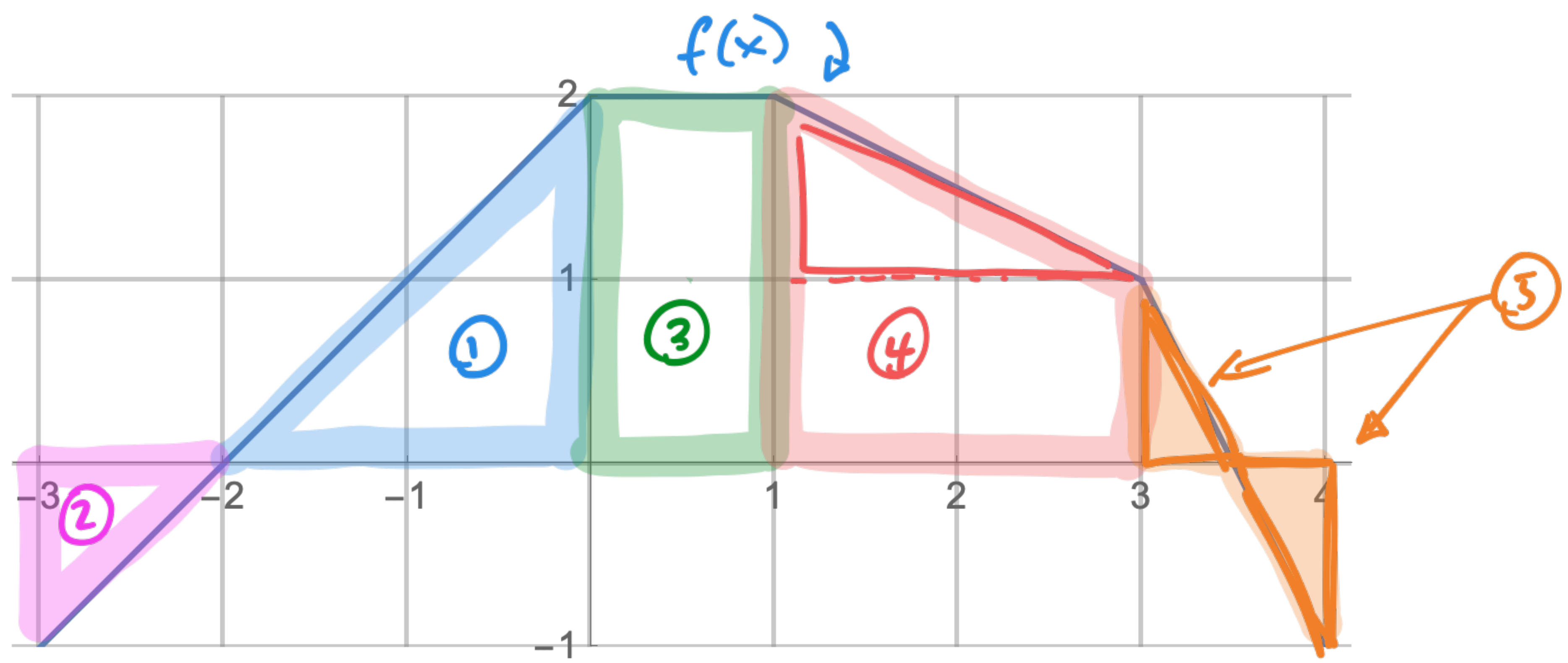
$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\cdot \Delta x$$

where $\Delta x = \frac{b-a}{n}$.

Example: write a limit equalling $\int_a^b x^3\,dx$

$$\int_a^b x^3\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n}}_{\int} \underbrace{\left(\frac{b-a}{n}\cdot i + a\right)^3}_{\underbrace{(\quad)^3}_{f(x)}} \underbrace{\left(\frac{b-a}{n}\right)}_{\substack{\Delta x \\ \downarrow \\ dx}}$$

Definite integrals and areas



1 $\int_{-2}^{0} f(x)dx = \frac{1}{2}(2)(2) = 2$

4 $\int_{1}^{3} f(x)dx = 2 + 1 = 3$

2 $\int_{-3}^{-2} f(x)dx = \frac{1}{2}(1)(-1) = -1/2$

5 $\int_{3}^{4} f(x)dx = 0$

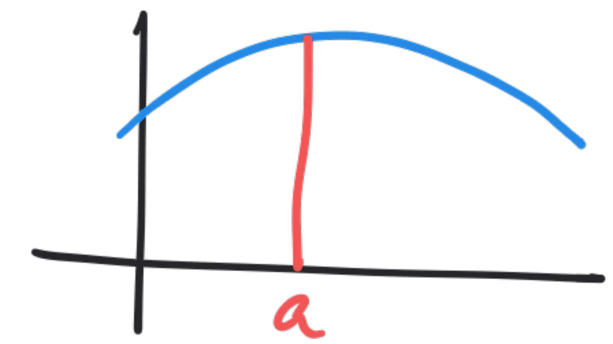
3 $\int_{0}^{1} f(x)dx = 2$

Entire integral $= \int_{-3}^{4} f(x)dx = \frac{-1}{2} + 2 + 2 + 3 + 0 = 6\frac{1}{2}$

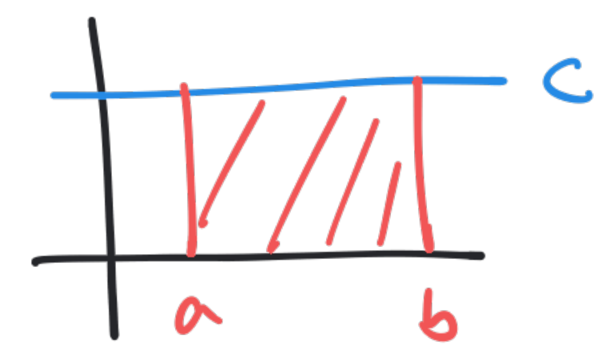
$\int_{-2}^{3} f(x)dx = 2 + 2 + 3 = 7$

Properties of definite integrals

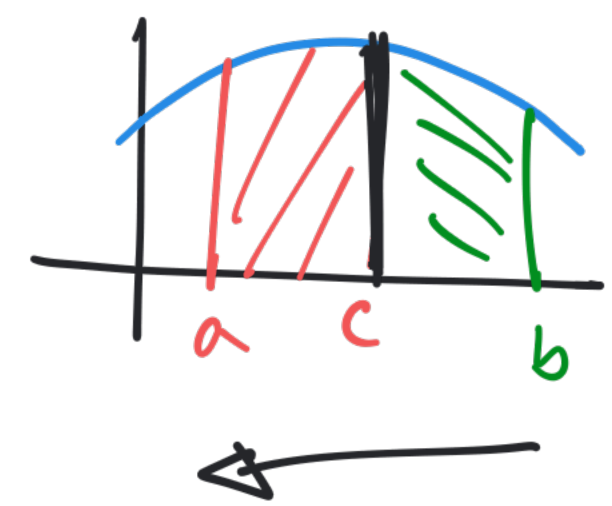
① $\displaystyle\int_a^a f(x)\,dx = 0$



② $\displaystyle\int_a^b c\,dx = (b-a)c$



③ $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



④ $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

⑤ $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

⑥ $\displaystyle\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$